Generalized Double Forward Derivatives

Parker Emmerson

June 2023

1 Calculus

Let $f(x)$ be a function and let $g(x)$ be its double forward derivative, then the future permutations of $f(x)$ can be mathematically expressed as:

$$f(x+h) = f(x) + hg(x) + \frac{h^2}{2!}\frac{d^2 f}{dx^2} + \mathcal{O}\left(h^3\right)$$

We can derive the above expression by using Taylor's theorem. By Taylor's theorem, we have:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \mathcal{O}\left(h^3\right)$$

Where $f'(x)$ is the first derivative and $f''(x)$ is the second derivative of $f(x)$. Substituting $f'(x)$ and $f''(x)$ with their respective forward derivatives $g(x)$ and $\frac{d^2 f}{dx^2}$, we get the desired expression:

$$f(x+h) = f(x) + hg(x) + \frac{h^2}{2!}\frac{d^2 f}{dx^2} + \mathcal{O}\left(h^3\right)$$

a) $\frac{d^2 f}{dx^2} = \frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$

One example of a rotational group applied to the double forward derivative is the rotation group SO (3) . This group allows for a change of the basis vectors in 3 - dimensional space, which affects the derivatives of a function . For example, if we consider the double forward derivative of a function $f\left(x_1, x_2, x_3\right)$ in the $\left(x_1, x_2, x_3\right)$ coordinate system, then the derivative can be written as :

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{k,l=1}^{3} \frac{\partial x^k}{\partial x_i}\frac{\partial x^l}{\partial x_j}\frac{\partial^2 f}{\partial x^k \partial x^l}$$

where $i, j, k, l = 1, 2, 3$. This equation shows how the double forward derivative of a function is affected by rotations in 3 - dimensional space. In terms of infinity tensor space, this equation implies that the derivatives of a function depend on the basis vectors of the space and how they are rotated. Map it onto the infinity tensor, where $g^\wedge\Omega[f]$ is the tensor' s order, $\zeta[f]$ is the weight function, $\kappa[f]$ is the factor of proportionality, and $\Omega[f]$ is the coefficient of proportionality. and show a few options for rotation. The infinity tensor can be written as:

$$\mathcal{M}_{g_{abcd\cdots fghij\ldots}} = g^{\Omega}(f)\zeta(f)\kappa(f)\Omega(f)\int_{\mathcal{N}} \mathrm{d}\theta \mathrm{d}s\ \mathrm{d}\delta \mathrm{d}\eta \rho g^{\Omega}(\theta)\frac{\partial x}{\partial \alpha}\frac{\partial}{\partial \theta}\frac{\partial}{\partial s}\frac{\partial}{\partial \delta}\frac{\partial}{\partial \eta}$$

The rotation group SO(3) can be applied to the infinity tensor by defining the basis vectors of the space as the rotation axes, such that :

$$\mathbf{x} = (x_1, x_2, x_3) \mapsto \mathbf{x}' = R(\theta)\mathbf{x} = (x_1', x_2', x_3')$$

where $R(\theta)$ is the rotation matrix. This means that the derivatives of the function $f\,(x_1, x_2, x_3)$ in the new coordinate system can be written as :

$$\frac{\partial^2 f}{\partial x_i' \partial x_j'} = \sum_{k,l=1}^{3} \frac{\partial^2 f}{\partial x^k \partial x^l}\frac{\partial x^k}{\partial x_i'}\frac{\partial x^l}{\partial x_j'}$$

Thus, the double forward derivative of a function can be affected by rotations in 3 - dimensional space.